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PARABOLIC CAPACITY AND SOBOLEV SPACES.(U)

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ABSTRACT

We prove here that, given an open subset Ω of $\mathbb{R}^N$, the usual parabolic capacity on $[0,T[\times \Omega$ associated with the heat operator $\frac{\partial}{\partial t} - \Delta$ can be defined using only the Hilbert norm of the space

$$W = \left\{ v \in L^2(0,T;H_0^1(\Omega));\ \frac{\partial v}{\partial t} \in L^2(0,T;H^{-1}(\Omega)) \right\}.$$

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SIGNIFICANCE AND EXPLANATION

In recent years, parabolic variational inequalities (V.I.) have been intensively developed in a functional analytic setting involving many function spaces. As in the case of elliptic V.I., the tools of potential theory have also proven to be most useful for solving and interpreting parabolic V.I. Several facts exhibit a close relationship between the functional analytic and potential theoretic approaches. Among them is the result provided in this paper. Let us describe its content.

Just as for the Laplacian operator, a capacity had been associated with the heat operator in order to solve various problems in potential theory. On the other hand, functional spaces - mainly Sobolev spaces, had been introduced to solve variational inequalities involving the heat operator. We prove here that this capacity can be defined in terms of the topology naturally induced by these functional spaces. This leads to interesting new results for parabolic variational inequalities.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

PARABOLIC CAPACITY AND SOBOLEV SPACES

Michel Pierre

INTRODUCTION

Let Ω be an open subset of $\mathbb{R}^N$ and $T > 0$. The usual parabolic capacity on $]0,T[\times\Omega$ associated with the heat operator $E = \frac{\partial}{\partial t} - \Delta$ is defined by

$$\forall\ \omega \subset]0,T[\times\Omega \quad \text{open}, \qquad c_0(\omega) = \int_{]0,T[\times\Omega} d\,E\,u_\omega\ ,$$

where u_ω is the capacitary potential of ω, that is the solution of the (formal) variational inequality:

$$(\mathrm{I}) \qquad \begin{cases} u \geq 1_\omega \text{ a.e.}, \quad u(0) = 1_\omega(0), \quad u(t,\cdot)|_{\partial\Omega} = 0 \\ \frac{\partial u}{\partial t} - \Delta u \geq 0\ , \ \frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{on } [u > 1_\omega]\ . \end{cases}$$

(Here 1_ω is the characteristic function of ω. Note that $Eu_\omega = \frac{\partial u_\omega}{\partial t} - \Delta u_\omega$ is a nonnegative measure on $]0,T[\times\Omega$). Another definition in terms of measures can also be found in [2].

We show in this paper that this capacity can be defined using only the Hilbert norm of the space:

$$\mathcal{W} = \{v \in L^2(0,T;H_0^1(\Omega))\ ;\ \frac{\partial v}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\}.$$

Namely, if we set, for any open subset ω of $]0,T[\times\Omega$:

$$c(\omega) = \inf\left(\|v\|_{\mathcal{W}}^2\ ;\ v \geq 1_\omega \text{ a.e. }\right\}\ ,$$

where

$$\|v\|_{\mathcal{W}}^2 = \|v\|^2_{L^2(0,T;H_0^1(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;H^{-1}(\Omega))}\ ,$$

then there exist $a,b > 0$ such that:

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(II) $\forall \omega$, $a \cdot c_0(\omega) \leq c(\omega) \leq b \cdot c_0(\omega)$.

It is well-known that this space $\mathcal{W}$ arises as the natural space of test-functions in numerous parabolic variational inequalities (V.I.) of type (I) (see Lions-Stampacchia [4], Lions-Magenes [5], Lions [3], Mignot-Puel [6] etc ...). On the other hand, as in the elliptic case, the tools of potential theory have also proven to be most useful to solve and interpret these parabolic V.I. (see [1],[8]). The above result emphasizes the strong relationship between the two approaches.

A direct consequence of (II) is that any element of $\mathcal{W}$ has a quasi-continuous representation. This fact (that we established in [8]) is an important tool to deduce fundamental properties about the structure of parabolic potentials (i.e. the functions $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ such that $\frac{\partial u}{\partial t} - \Delta u \geq 0$) (see [8], [10] for these results).

Another consequence is that, as in the elliptic case, "L^2-estimates" can be used to evaluate the parabolic capacity of a set. In the same spirit, we also show here the following result: if u is a parabolic potential greater than or equal to 1 on ω, then the capacity of ω can be estimated by the norm of u in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Lastly, this suggests that for the nonlinear problems associated with operators of the form

$$\frac{\partial u}{\partial t} - \operatorname{div} A(x,u,Du),$$

the natural capacity can be defined by the norm of

$$\mathcal{W}_p = \{v \in L^p(0,T;W^{1,p}(\Omega));\ \frac{\partial u}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega))\}$$

where $p \in]1,\infty[$ is suitably chosen and $\frac{1}{p} + \frac{1}{p'} = 1$.

In this paper, we state our result in the general setting of Dirichlet parabolic spaces so that it can be applied to general elliptic operators with Dirichlet, Neumann or mixed boundary conditions.

1°). Parabolic Dirichlet space

Let X be a locally compact space, countable at the infinity,[†] ξ a Radon measure on X whose support is X. We denote $K(X)$ (resp. $K^+(X)$) the space of continuous (resp. nonnegative and continuous) real functions with compact support in X. The space $K(X)$ is equipped with its usual locally convex topology.

Let V be a Hilbert space with the norm $\|\cdot\|$; we assume that V is embedded into $L^2(X)$, the space of (classes of) real square integrable functions with the norm

$$|u|_2 = \left[\int_X u^2(x)\,d\xi(x)\right]^{1/2}.$$

Then, if V' is the dual space of V, we have

(1) $\qquad V \hookrightarrow L^2(X) \hookrightarrow V'$.

The scalar product in $L^2(X)$ as well as the duality (V',V) will be denoted by $(\cdot,\cdot)$.

We will assume:

(2) $\qquad K(X) \cap V$ is dense in V and $K(X)$.

Example 1. $\quad(\alpha)$ $X = \mathbf{R}^N$, $V = H^1(\mathbf{R}^N)$, $V' = H^{-1}(\mathbf{R}^N)$.

(β) $\quad X = \Omega$ open set in $\mathbf{R}^N$, $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$.

(γ) $\quad X = \bar{\Omega}$, $V = H^1(\Omega)$ (Ω regular bounded open set in $\mathbf{R}^N$).

(δ) $\quad X = \{1 \text{ point}\}$, $V = L^2(X) = \mathbf{R}$.

Given $T > 0$, we denote $Q = [0,T[\times X$ equipped with the Radon measure $dt \otimes \xi$ where dt is the Lebesgue measure on $[0,T[$. $K(Q)$ will denote the space of continuous numerical functions with compact support in Q , equipped with its natural topology.

Now, associated with V , V', we have

$$\mathcal{V} = L^2(0,T;V) \text{ and its dual } \mathcal{V}' = L^2(0,T;V').$$

$$\mathcal{W} = \left\{v \in \mathcal{V} ;\ \frac{dv}{dt} \in \mathcal{V}'\right\} .$$

These spaces are Hilbert spaces with the norms:

$$\|v\|_{\mathcal{V}}^2 = \int_0^T \|v(t)\|^2 dt, \quad \|v\|_{\mathcal{V}'}^2 = \int_0^T \|v(t)\|_{V'}^2 dt , \quad \|v\|_{\mathcal{W}}^2 = \|v\|_{\mathcal{V}}^2 + \left\|\frac{\partial v}{\partial t}\right\|_{\mathcal{V}'}^2 .$$

Let us recall that $\mathcal{W}$ is embedded into $C([0,T[;\ L^2(X))$ (see Lions-Magenes [5]).

† That is X is the union of a countable number of compact subsets.

As a consequence of (2), one can show that (see [8]):

(3) $\quad \mathcal{K}(Q) \cap \mathcal{W}$ is dense in $\mathcal{W}$ and $\mathcal{K}(Q)$.

<u>The operators $A(t)$.</u>

For a.e. t, let $a(t,\cdot,\cdot)$ be a bilinear form on $V \times V$ satisfying:

(4) $\quad \forall\, u,v \in V \times V\,,\ t \mapsto a(t,u,v)$ is measurable

(5) $\quad \exists\, M > 0\,,\ \forall (u,v) \in V \times V\,,\ \text{a.e. } t \in (0,T),\ |a(t,u,v)| \le M\|u\| \cdot \|v\|$

(6) $\quad \exists \alpha > 0\,,\ \forall\, v \in V\,,\ \text{a.e. } t \in (0,T),\ a(t,v,v) \ge \alpha\|v\|^2$.

With $a(t,\cdot,\cdot)$ and its adjoint $a^*(t,u,v) = a(t,v,u)$ are associated two continuous operators from V into V' defined by

$$\forall\, u,v \in V\,,\quad (A(t)u,v) = a(t,u,v),\quad (A^*(t)u,v) = a^*(t,u,v).$$

We will also assume that $A(t)$ and $A^*(t)$ satisfy maximum principle properties, namely that the contractions $r \mapsto |r|$ and $r \mapsto r^+ \wedge 1$ operate on V equipped with a and a^* that is:

(7) $\quad \forall\, v \in V\,,\ v^+ \in V,\ v^- \in V$ and a.e. $t \in (0,T)$, $a(t,v^+,v^-) \ge 0$.

(8) $$\begin{cases} \forall\, v \in V\,,\ v^+ \wedge 1 \in V \text{ and} \\ \text{a.e. } t \in (0,T),\ a(t, u + u^+ \wedge 1, u - u^+ \wedge 1) \ge 0 \\ \qquad\qquad a(t, u - u^+ \wedge 1, u + u^+ \wedge 1) \ge 0\,. \end{cases}$$

<u>Examples 2</u>. Corresponding to the choices of X and V in the examples above one can successively choose:

$$(\alpha)\qquad a(t,u,v) = \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} a_{ij}(x,t)\,\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_j}\,dx + \sum_{i=1}^{N} \int_{\mathbb{R}^N} b_i(x,t)\,\frac{\partial u}{\partial x_i}\,v\,dx$$

$$+ \int_{\mathbb{R}^N} c_0(x,t)\,u\,v\,dx\,,$$

where $a_{ij}, b_i, c_0 \in L^\infty(]0,T[\times \mathbb{R}^N)$ and satisfy

$$\exists\, \alpha > 0\,,\ \forall\, \xi \in \mathbb{R}^N,\ \sum_{i,j=1}^{N} a_{ij}\,\xi_i \xi_j \ge \alpha\Big(\sum_{i=1}^{N} \xi_i^2\Big) \text{ a.e. on } Q\,.$$

Then, $a(\cdot,\cdot,\cdot)$ satisfies (4) and (5). It satisfies (7) and (8) if $c_0 \ge 0$ and satisfies (6) if $c_0 \ge \lambda$ for λ large enough. Since we will study parabolic properties, the latter point is not a restriction.

(β), (γ) One can choose $a(\cdot,\cdot,\cdot)$ as above where one replaces $\mathbf{R}^N$ by Ω.

(δ) Take a defined by

$$\text{a. e. } t \in (0,T), \quad \forall u,v \in \mathbf{R}, \quad a(t,u,v) = a(t)uv,$$

where $a \in L^\infty(0,T)$, $a > 0$.

<u>Parabolic potentials.</u>

<u>Definition 1.</u> We shall call <u>parabolic potential</u> any element of

$$P = \Big\{u \in L^2(0,T;V) \cap L^\infty(0,T;L^2(X)); \quad \forall v \in W \text{ with } v(T) = 0, \quad v > 0,$$
$$\int_0^T \Big[\Big(-\frac{\partial v}{\partial t}(t), u(t)\Big) + a(t,u(t), v(t))\Big]dt > 0\Big\}.$$

<u>Remark.</u> We will often omit the variable t in the integral above and write it as

$$\int_0^T \Big(-\frac{\partial v}{\partial t}, u\Big) + a(u,v).$$

Thanks to Hahn-Banach theorem, we have (see [8],[10]):

<u>Proposition 1.</u> Let $u \in P$; then there exists a unique Radon measure on Q, denoted Eu, such that

$$\forall v \in W \cap K(Q) \text{ with } v(T) = 0,$$
$$\int_0^T \Big(-\frac{\partial v}{\partial t}, u\Big) + a(u,v) = \int_Q v \, d(Eu).$$

Details are given in [8], [10] about the space P and the measures Eu. Let us just make them explicit in a particular but typical example.

<u>Example 3</u>. Let $X = \Omega$, $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$ and

$$\forall t \in [0,T], \quad \forall u,v \in V, \quad a(t,u,v) = \int_\Omega \nabla u \, \nabla v.$$

Then, if $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$,

$$(u \in P) \Leftrightarrow \Big(u > 0, \quad \frac{\partial u}{\partial t} - \Delta u > 0 \quad \text{in } \mathcal{D}'(]0,T[\times \Omega)\Big).$$

Moreover,

$$Eu = u(0^+)dx_0 + \frac{\partial u}{\partial t} - \Delta u,$$

where dx_0 is the Lebesgue measure induced on $\{0\} \times \Omega$ and

$u(0^+) = \operatorname{ess} \lim_{t \to 0^+} u(t)$ in $L^2(\Omega)$.

More examples are given in [8].

2°) The main result.

Let us first recall the usual definition of the parabolic capacity associated with the operators $A(t)$.

For any open set ω of Q, we consider

$$\mathcal{P}_\omega = \{u \in \mathcal{P} \,;\; u \geq 1 \text{ a.e. on } \omega\}$$

Then, if $\mathcal{P}_\omega$ is not empty, it has a smallest element u_ω called the capacitary potential of ω (see [8], [10] for a proof).

Definition 1: For any open set $\omega \subset Q$, we set

$$c_0(\omega) = \begin{cases} \int_Q d\, \mathcal{E}u_\omega & \text{if } \mathcal{P}_\omega \neq \emptyset \\ +\infty & \text{if } \mathcal{P}_\omega = \emptyset . \end{cases}$$

For any $E \subset Q$, we define:

$$\text{capacity of } E = c_0(E) = \inf_{\substack{\omega \supset E \\ \omega \text{ open}}} c_0(\omega) .$$

Now let us define two different capacities. For that, we denote Λ the space $\mathcal{V} \cap L^\infty(0,T;L^2(X))$ with the norm:

$$\|u\|_\Lambda^2 = \|u\|_{\mathcal{V}}^2 + \sup_{t \in (0,T)} \text{ess}\, |u(t)|_2^2 .$$

Definitions 2 and 3. For any open set $\omega \subset Q$, we set:

$$c_1(\omega) = \inf\{\|u\|_\Lambda \,;\; u \in \mathcal{P},\; u \geq 1 \text{ a.e. on } \omega\}$$

$$c_2(\omega) = \inf\{\|v\|_{\mathcal{W}} \,;\; v \in \mathcal{W}^+,\; v \geq 1 \text{ a.e. on } \omega\} .$$

For any $E \subset Q$, we define:

$$c_1(E) = \inf_{\substack{\omega \supset E \\ \omega \text{ open}}} c_1(\omega) , \qquad c_2(E) = \inf_{\substack{\omega \supset E \\ \omega \text{ open}}} c_2(\omega) .$$

Then, we have the main result.

Theorem 1. There exist $a,b > 0$ such that, for any $E \subset \Omega$:

$$\text{(i)} \quad a \cdot c_0(E) \leq [c_1(E)]^2 \leq b \cdot c_0(E)$$

$$\text{(ii)} \quad a \cdot c_0(E) \leq [c_2(E)]^2 \leq b \cdot c_0(E) .$$

Remarks. According to this result, to estimate the parabolic capacity of a set E , one can

(i) Find $u \in \mathcal{P}$ with $u \geq 1$ on a neighborhood of E and compute the Λ-norm of u , or

(ii) Find $v \in \mathcal{W}$ with $v \geq 1$ on a neighborhood of E and compute the $\mathcal{W}$-norm of v .

Note that the definition of $c_1(\cdot)$ still involves $\mathcal{P}$ and hence the operators $A(t)$, but it uses the Hilbert-norms of $\mathcal{V}$ and $L^2(X)$ instead of an "L^1-norm" as in the definition of $c_0(\omega)$.

The interest of the definition of $c_2(\cdot)$ is that it only involves the topology of the Hilbert space $\mathcal{W}$ and does not depend on the operators $A(t)$.

Recall that $\mathcal{W} \hookrightarrow \Lambda$; so the topology of Λ is weaker than the topology of $\mathcal{W}$. But it is also sufficient to estimate the capacity of a set if one uses elements of $\mathcal{P}$.

If $c_1(\cdot)$ and $c_2(\cdot)$ are not generally "strong" capacities, they are however "weak" capacities. Namely:

Proposition 2.

(i) For $i = 0,1,2,$

(a) $E_1 \subset E_2 \Rightarrow c_i(E_1) \leq c_i(E_2)$.

(b) For any nondecreasing sequence (E_n) of subsets of Ω

$$c_i\left(\bigcup_n E_n\right) = \sup_n c_i(E_n) .$$

(c) For any nonincreasing sequence (K_n) of compacts of Ω

$$c_i\left(\bigcap_n K_n\right) = \inf_n c_i(K_n).$$

(ii) (Strong subadditivity) $\forall E_1,E_2 \subset \Omega$,

$$c_0(E_1 \cup E_2) + c_0(E_1 \cap E_2) \leq c_0(E_1) + c_0(E_2).$$

(iii) ("Weak" subadditivity) For $i = 1,2$, $\forall E_1, E_2 \subset Q$,

$$c_i(E_1 \cup E_2) \leq c_i(E_1) + c_i(E_2).$$

The properties of $c_0(\ )$ have already been studied in [8] (or [10]); we shall not reproduce the proofs here.

Only the property (b) is difficult for $c_1(\cdot)$ and $c_2(\cdot)$. It will result from important properties of the spaces P and W that will also be used to prove the part (ii) of Theorem 1. But let us begin by the proof of (i) in Theorem 1 which is fairly easy.

Proof of (i) in Theorem 1.

It is sufficient to prove it for any open set $\omega \subset Q$.

Let us prove that, if $P_\omega \neq \phi$;

(9) $$\|u_\omega\|^2_\Lambda \leq (2 + \frac{1}{\alpha})\ c_0(\omega).$$

In order to compute, we need to approximate u_ω by more "regular" potentials. This is the purpose of the Theorem I-1 in [8] which says that the solution of:

(10) $$u_\lambda \in W,\ u_\lambda(0) = u_\omega(0),\ u_\lambda + \lambda(\frac{\partial u_\lambda}{\partial t} + A\,u_\lambda) = u_\omega \quad (\lambda > 0),$$

satisfies

$$u_\lambda \in P\ ,\ u_\lambda \leq u_\omega\ ,\ \int_Q d\ Eu_\lambda \leq \int_Q d\ E\ u_\omega\ ,$$

and converges in $L^2(0,T;L^2(X))$ and weakly in V to u_ω when $\lambda \to 0^+$. But for any $t \in (0,T)$:

$$\tfrac{1}{2}\,|u_\lambda(t)|_2^2 + \tfrac{1}{2}\,|u_\lambda(0)|_2^2 + \int_0^t a(u_\lambda,u_\lambda) = \int_0^t (\frac{\partial u_\lambda}{\partial t} + Au_\lambda, u_\lambda) + (u_\lambda(0),\ u_\lambda(0)).$$

Since $0 \leq u_\lambda \leq u_\omega \leq 1$, the right-hand side (which is formally equal to $\int_{[0,t]\times X} u_\lambda\ d\ Eu_\lambda)$ is less than $\int_Q d\ Eu_\lambda$ (see [8] prop. I-3). Hence, for any λ, by (8):

$$\tfrac{1}{2}\,|u_\lambda(t)|_2^2,\quad \alpha\|u_\lambda\|^2_V \leq \int_Q d\ Eu_\lambda \leq c_0(\omega)\ .$$

Letting λ go to 0 gives (9) and the second inequality of(i) with $b = 2 + \frac{1}{\alpha}$.

For the first inequality, let $\omega \subset Q$ open and $u \in P$ with $u \geq 1$ a.e. on ω. For any compact $K \subset \omega$, there exists $\Psi \in K(Q) \cap W^+$ equal to 1 on K and with support in ω (see [8], Lemma II-2). Then, if u_K is the capacitary potential of K, $E u_K$ is carried by K (see [8], [10]). Therefore:

$$(11) \qquad c_0(K) = \int_Q d\, E u_K \leq \int_Q \Psi \, d E u_K .$$

Now, if u_λ is the solution of (10) where u_ω is replaced by u_K, since $\frac{\partial u_\lambda}{\partial t} + A u_\lambda \geq 0$ and $\psi \leq u$, we have:

$$\int_Q \psi \, d E u_\lambda = (\psi(0), u_\lambda(0)) + \int_0^T \left(\frac{\partial u_\lambda}{\partial t} + A u_\lambda, \psi \right)$$

$$\leq (u(0), u_\lambda(0)) + \int_0^T \left(\frac{\partial u_\lambda}{\partial t} + A u_\lambda, u \right).$$

Using $u \in P$, we obtain:

$$\int_Q \psi \, d E u_\lambda \leq (u(0), u_\lambda(0)) + (u(T), u_\lambda(T)) + \int_0^T a(u, u_\lambda) + a(u_\lambda, u) .$$

When λ goes to 0^+, $E u_\lambda$ converges to Eu in the sense of measure. Hence, using (11), we have:

$$(12) \qquad c_0(K) \leq |u(0)|_2 |u_K(0)|_2 + |u(T)|_2 |u_K(T)|_2 + \int_0^T a(u, u_K) + a(u_K, u) .$$

But if $P_\omega \neq \phi$ there exists a nondecreasing sequence of compacts $K_n \subset \omega$ such that $c_0(K_n)$ converges to $c_0(\omega)$ and u_{K_n} weakly converges to u_ω in V (see for instance [8] Prop. II-4). Then, passing to the limit in (12), we obtain that there exists c depending only on M (see (5)) such that:

$$c_0(\omega) \leq c \|u\|_\Lambda \|u_\omega\|_\Lambda .$$

This together with (9) completes the proof of (i) in Theorem 1.

<u>Proof of (ii) in Theorem 1.</u>

It is a direct consequence of the part (i) and the following proposition.

Proposition 3. There exists $k > 0$ such that

(i) $\forall u \in \mathcal{K}$, $\exists v \in \mathcal{W}$ with

$$v \geq u \ , \quad \|v\|_{\mathcal{W}} \leq k\|u\|_{\Lambda} \ .$$

(ii) $\forall v \in \mathcal{W}$, $\exists u \in \mathcal{K}$ with

$$u \geq v^{+} \ , \quad \|u\|_{\Lambda} \leq k\|v\|_{\mathcal{W}} \ .$$

Proof of Proposition 3.

For (i), given $u \in \mathcal{K}$, we consider the solution v of:

$$v \in \mathcal{W}, \quad v(T) = u(T^{-}), \quad -\frac{\partial v}{\partial t} + A^{*}(t)v = A^{*}(t)u + A(t)u \ . \tag{13}$$

By well-known results about these linear parabolic equations (see Lions-Magenes [5]), such a solution exists in $\mathcal{W}$ and there exists a constant c depending only on $A(t)$ such that:

$$\|v\|_{\mathcal{W}} \leq c\,[\,|u(T)|_2 + \|A^{*}(t)u\|_{V'} + \|A(t)u\|_{V'}\,] \ .$$

That is:

$$\|v\|_{\mathcal{W}} \leq k\|u\|_{\Lambda} \ ,$$

where k depends only on $A(t)$. Moreover, we formally have:

$$-\frac{\partial}{\partial t}(v-u) + A^{*}(t)(v-u) = \frac{\partial u}{\partial t} + A(t)u \geq 0 \quad (\text{since } u \in \mathcal{K}).$$

Since $(v-u)(T) = 0$, by the maximum principle, $v \geq u$. This formal computation can be justified in the following way. Given $f \in L^2(0,T;L^2(X))$, $f \geq 0$, let us consider the solution w of:

$$w \in \mathcal{W}, \quad w(0) = 0, \quad \frac{\partial w}{\partial t} + A(t)w = f \ .$$

By the maximum principle (see (7)), $f \geq 0 \Rightarrow w \geq 0$. But

$$\int_0^T \left(\frac{\partial w}{\partial t} + A(t)w, v\right) = (v(T), w(T)) + \int_0^T \left(-\frac{\partial v}{\partial t} + A^{*}(t)v, w\right) \ .$$

This implies

$$\int_0^T (f, v-u) = (u(T), w(T)) + \int_0^T \left(-\frac{\partial w}{\partial t}, u\right) + a(u,w) \ .$$

Since $w \geq 0$ and $u \in \mathcal{K}$, the right-hand side is nonnegative. As f is arbitrary, this implies $v \geq u$.

For (ii), given $v \in \mathcal{W}$, we consider

$$u = \inf\{w \in \mathcal{K};\ w \geq v\} = \inf\{w \in \mathcal{K};\ w \geq v^{+}\} \ . \tag{14}$$

Using the results of Mignot-Puel [6], it can be shown (see also [8] Lemma II-1) that $u \in \mathcal{P}$ and is the limit in $L^2(0,T;L^2(X))$ and weakly in $\mathcal{V}$ of the solution u_ε of the penalized problem

$$u_\varepsilon \in \mathcal{P},\quad u_\varepsilon(0) = v(0),\quad \frac{\partial u_\varepsilon}{\partial t} + A(t)u_\varepsilon - \frac{1}{\varepsilon}(u_\varepsilon - v)^- = 0 \quad (\varepsilon > 0)\ .$$

But, for any $t \in (0,T)$:

$$\tfrac{1}{2}\,|u_\varepsilon(t)|_2^2 - \tfrac{1}{2}\,|v(0)|_2^2 + \int_0^t a(u_\varepsilon,u_\varepsilon) = \int_0^t \Big(\frac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon, u_\varepsilon\Big)$$

$$= \int_0^t \Big(\frac{1}{\varepsilon}\,(u_\varepsilon - v)^-, u_\varepsilon - v\Big) + \int_0^t \Big(\frac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon, v\Big)$$

$$\leq (u_\varepsilon(t), v(t)) - (v(0), v(0)) + \int_0^t \Big(-\frac{\partial v}{\partial t} + A^* v, u_\varepsilon\Big)\ .$$

Passing to the limit gives

$$\tfrac{1}{2}\,|u(t)|_2^2 + \alpha\|u\|_{\mathcal{V}}^2 \leq |u(t|_2|v(t)|_2 + \Big\| -\frac{\partial v}{\partial t} + A^* v\Big\|_{\mathcal{V}'} \cdot \|u\|_{\mathcal{V}}\ .$$

Hence, there exists a constant k depending only on $A(t)$ such that:

$$\|u\|_\Lambda^2 \leq k\ \|v\|_{\mathcal{W}} \cdot \|u\|_\Lambda\ .$$

Since $u \in \mathcal{P}$ and $u \geq v^+$, this completes the proof.

In order to prove the Proposition 2, let us introduce for any $E \subset Q$:

$$\mathcal{W}_E = \{v \in \mathcal{W}^+;\ v = \lim_{n\to\infty} v_n \ \text{ in } \mathcal{W} \text{ with } v_n \geq 1 \text{ a.e. on a neighbourhood of } E\}.$$

$$\mathcal{P}_E = \{u \in \mathcal{P};\ \exists u_n \in \mathcal{P} \text{ with } u = \lim u_n \text{ in } \mathcal{V},\ \limsup_{n\to\infty}\|u_n\|_\Lambda \leq \|u\|_\Lambda,$$
$$u(T) = \lim_{n\to\infty} u_n(T) \text{ in } L^2(X) \text{ and } u_n \geq 1 \text{ on a neighbourhood of } E\}.$$

If $E = \omega$ is an open set, we immediately have:

$$\mathcal{W}_\omega = \{v \in \mathcal{W}^+;\ v \geq 1 \ \text{ a.e. on } \omega\}.$$

$$\mathcal{P}_\omega = \{u \in \mathcal{P};\ u \geq 1 \ \text{ a.e. on } \omega\}.$$

Moreover, we verify that, for any $E \subset Q$:

$$c_1(E) = \inf\{\|u\|_\Lambda;\ u \in \mathcal{P}_E\}$$

$$c_2(E) = \inf\{\|v\|_{\mathcal{W}};\ v \in \mathcal{W}_E\}\ .$$

Remark that $\mathcal{W}_E$ is a closed convex set in $\mathcal{W}$. Hence, if v_E is the projection of 0 on $\mathcal{W}_E$ in the Hilbert space $\mathcal{W}$, then $c_2(E) = \|v_E\|_{\mathcal{W}}$.

Lemma 1. For any nondecreasing sequence (E_n) of subsets of Ω :

$$\text{(i)} \quad \bigcap_n \mathcal{W}_{E_n} = \mathcal{W}_{\bigcup_n E_n}$$

$$\text{(ii)} \quad \bigcap_n \mathcal{P}_{E_n} = \mathcal{P}_{\bigcup_n E_n}$$

To prove Lemma 1, we will need the following consequence of the Proposition 3:

Lemma 2. There exists $k > 0$ such that, for any $v \in \mathcal{W}$, there exists $w \in \mathcal{W}$ with:

$$w \geq |v|, \ \|w\|_{\mathcal{W}} \leq k\|v\|_{\mathcal{W}} .$$

Proof of Lemma 2.

Let $v \in \mathcal{W}$, by (ii) in Proposition 3, there exist $u_1, u_2 \in \mathcal{P}$ such that

$$u_1 \geq v^+ , u_2 \geq v^- , \|u_1\|_\Lambda , \|u_2\|_\Lambda \leq k\|v\|_{\mathcal{W}} .$$

Now by (i) of the same proposition, there exists $w \in \mathcal{W}$ with

$$w \geq u_1 + u_2 , \ \|w\|_{\mathcal{W}} \leq k\|u_1 + u_2\|_\Lambda .$$

Then, $w \geq v^+ + v^- = |v|$ and satisfied

$$\|w\|_{\mathcal{W}} \leq 2k^2\|v\|_{\mathcal{W}} .$$

Remark. As a consequence of (7), if $v \in V$, then v^+, v^- and $|v|$ also belong to V and the norm of $|v|$ in V can be estimated in terms of the norm of v.

But, there is no such estimate in $\mathcal{W}$ (see L. Tartar's remark in appendix). However Lemma 2 will be sufficient for our purpose.

Proof of Lemma 1.

Let $E = \bigcup_n E_n$; the inclusions $\mathcal{W}_E \subset \bigcap_n \mathcal{W}_{E_n}$, $\mathcal{P}_E \subset \bigcap_n \mathcal{P}_{E_n}$ are obvious.

Let $v \in \bigcap_n \mathcal{W}_{E_n}$; then there exists $v_n \in \mathcal{W}$ with $v_n \geq 1$ on a neighborhood ω_n of E_n and $\|v - v_n\|_{\mathcal{W}} \leq 2^{-n}$. The serie $\sum_1^\infty (v_{n+1} - v_n)$ is converging in $\mathcal{W}$. By Lemma 2, there exists $w_n \in \mathcal{W}$ with

$$w_n \geq |v_{n+1} - v_n|, \ \|w_n\|_{\mathcal{W}} \leq k\|v_{n+1} - v_n\|_{\mathcal{W}} .$$

Hence the serie $\sum_1^\infty w_n$ is converging in $\mathcal{W}$.

Now set $q_n = v_n + \sum_n^{+\infty} w_k$. If $k > n$:

$$g_n \geq v_n + \sum_n^{k-1} w_j \geq v_n + \sum_n^{k-1} (v_{j+1} - v_j) = v_k \geq 1 \quad \text{a.e. on } \omega_k .$$

Hence, $g_n \geq 1$ a.e. on $\bigcup_{n+1}^{\infty} \omega_k$ which is a neighborhood of E and $v = \lim g_n$ in $\mathcal{W}$. Therefore $v \in \mathcal{W}_E$.

Now let $u \in \bigcap_n P_{E_n}$; then there exists $u_n \in P$ such that $\|u - u_n\|_V + |u(T) - u_n(T)|_2 \leq \frac{1}{n}$ and $u_n \geq 1$ on a neighborhood ω_n of E_n. For any $\lambda > 0$, we consider the solution of

$$\begin{cases} v_n^\lambda \in \mathcal{W} \ , \quad v_n^\lambda(T) = u_n(T) \ , \\ v_n^\lambda + \lambda\left(-\dfrac{\partial v_n^\lambda}{\partial t} + A^* v_n^\lambda\right) = u_n + \lambda(Au_n + A^* u_n). \end{cases}$$

Then, by [8] Lemma IV-1, $v_n^\lambda \geq u_n$. (Remark that formally $v_n^\lambda - u_n + \lambda\left(-\frac{\partial}{\partial t}(v_n^\lambda - u_n) + A^*(v_n^\lambda - u_n)\right) = \lambda\left(\frac{\partial u_n}{\partial t} + Au_n\right) \geq 0$). Moreover, for λ fixed, v_n^λ converges in $\mathcal{W}$ to the solution of

$$\begin{cases} v^\lambda \in \mathcal{W}, \quad v^\lambda(T) = u(T) \\ v^\lambda + \lambda\left(-\dfrac{\partial v^\lambda}{\partial t} + A^* v^\lambda\right) = u + \lambda(Au + A^* u). \end{cases}$$

Indeed:

$$\|v_n^\lambda - v^\lambda\|_{\mathcal{W}} \leq c_\lambda(\|u_n - u\|_V + |u_n(T) - u(T)|_2).$$

Since $v_n^\lambda \geq u_n \geq 1$ on ω_n, as in the proof of Lemma 2, for any $\lambda > 0$, we can construct $g_n^\lambda \in \mathcal{W}$ converging in $\mathcal{W}$ to v^λ with $g_n^\lambda \geq 1$ on a neighborhood of E. Let us choose $g_\lambda = g_{n_\lambda}^\lambda$ such that $\|g_\lambda - v^\lambda\|_{\mathcal{W}} \leq \lambda$.

By Proposition 3, there exists $u_\lambda \in P$ with $u_\lambda \geq g_\lambda - v^\lambda$ and $\|u_\lambda\|_\Lambda \leq k\|g_\lambda - v^\lambda\|_{\mathcal{W}} \leq k\lambda$. Moreover, by the results in [8], Section IV, there exists a convex combination of the v^λ (still denoted by v^λ) such that:

- v^λ converges to u in V
- $\lim_{\lambda \to \infty} \| v^\lambda\|_\Lambda = \|u\|_\Lambda$
- if $\hat{u}_\lambda = \inf\{u \in P \ ; \ u \geq v^\lambda\}$, $\hat{u}_\lambda - v^\lambda$ converges to 0 in Λ.

Then, $u_\lambda \to$ [illegible], $u_\lambda \to u_\lambda$ [illegible] on a neighborhood of F, $u_\lambda \to$ [illegible] converges to u in [illegible], $u_\lambda(T) \to u_\lambda(T)$ converges to $u(T)$ in $L^2(X)$ and $\lim_{\lambda \to 0}$ [illegible] $= u$.

Hence u [illegible]$_E$.

<u>Proof of Proposition 2.</u>

The properties of [illegible]$_0(\cdot)$ are shown in [illegible]. The part (a) of (i) is obvious. The point (b) is a direct consequence of Lemma 1.

For (c), remark that, for $i = 1,2$

$$c_i(\cap K_n) \leqslant \inf_n c_i(K_n) .$$

Now, for $\varepsilon > 0$, there exists a neighborhood ω_ε of $K = \cap K_n$ such that

$$c_1(\omega_\varepsilon) \leqslant c_1(K) + \varepsilon , \qquad c_2(\omega_\varepsilon) \leqslant c_2(K) + \varepsilon .$$

But as K_n is a sequence of compacts decreasing to K, for n large enough, $K_n \subset \omega_\varepsilon$. Hence:

$$\inf_n c_i(K_n) \leqslant c_i(K_n) \leqslant c_i(\omega_\varepsilon) \leqslant c_i(K) + \varepsilon .$$

For (iii), we use the subadditivity of $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_\Lambda$.

3°) Application.

We proved in [8] that the elements of $\mathcal{W}$ are quasi-continuous. We will give here a more direct proof using essentially the equivalent definition of the capacity given by Theorem 1 in terms of the $\mathcal{W}$-norm, together with Lemma 2. (See also [7] for abstract "elliptic" results of this kind).

We recall that, given a capacity $c(\cdot)$ on Q :

Definition. A function $v : Q \to \mathbb{R}$ is said to be quasi-continuous if there exists a nonincreasing sequence of open set $\omega_n \subset Q$ with

(i) $\lim\limits_{n\to\infty} c(\omega_n) = 0$

(ii) the restriction of v to the complement of ω_n is continuous for all n .

Remark. This definition is clearly invariant when one replaces $c(\cdot)$ by an " equivalent" capacity $\hat{c}(\cdot)$, that is a capacity satisfying for some $\alpha > 0$:

$$a,b > 0 \ , \quad E \subset Q \ , \quad a\cdot c(E) \leq [\hat{c}(E)]^{\alpha} \leq b\cdot c(E) \ .$$

Hence the notion of quasi-continuity is the same for our capacities $c_0(\cdot)$, $c_1(\cdot)$ and $c_2(\cdot)$.

Theorem 2. Any element v of $\mathcal{W}$ has a unique quasi-continuous representation $\tilde{v}$.

Remark. "Unique" means here that, if $\hat{v}$ is quasi-continuous and satisfies $\hat{v} = \tilde{v}$ a.e., then $\hat{v} = \tilde{v}$ quasi - everywhere (i.e. everywhere except on a set of zero capacity).

Proof of Theorem 2.

Let $v \in \mathcal{W}$; by density of $K(Q) \cap \mathcal{W}$ in $\mathcal{W}$, there exist $v_n \in \mathcal{W} \cap K(Q)$ converging to v with

$$\sum_{n=1}^{\infty} 2^n \|v_{n+1} - v_n\|_{\mathcal{W}} < +\infty \ .$$

Let $\omega_n = \{z \in Q;\ |v_{n+1}(z) - v_n(z)| > 2^{-n}\}$ and $\Omega_p = \bigcup_{n\geq p} \omega_n$.

By Lemma 2, there exists $w_n \in \mathcal{W}$ with

$$w_n \geq |v_{n+1} - v_n| \ , \quad \|w_n\|_{\mathcal{W}} \leq k \cdot \|v_{n+1} - v_n\|_{\mathcal{W}} \ .$$

Hence

$$c_2(\omega_n) \leq c_2(\{z \in Q \ ; \ w_n(z) > 2^{-n}\}) \leq 2^n \|w_n\|_{\mathcal{W}} \ .$$

This proves that $\lim_{n\to\infty} c_2(\omega_n) = 0$. But, for any n:

$$|v_{n+1}(z) - v_n(z)| \leq 2^{-n} \quad \forall z \notin \Omega_p \ , \quad \forall n \geq n \ .$$

Hence, v_n converges uniformly on the complement of each Ω_p. The limit $\tilde{v}$ is defined quasi-everywhere (everywhere except on $\cap_p \Omega_p$ which is of zero capacity), $\tilde{v}$ is quasi-continuous and $\tilde{v} = v$ a.e. .

For the uniqueness, let us consider $\hat{v}$ quasi-continuous with $\tilde{v} = \hat{v}$ a.e. and ω_n a sequence of open sets associated with $\tilde{v} - \hat{v}$ (see the definition above). Then, $A_n = \{z \in Q;\ \tilde{v} - \hat{v} < 0\} \cup \omega_n$ is open for any n. Since $\{z \in Q;\ \tilde{v} - \hat{v} < 0\}$ is of measure 0 , $\mathscr{K}_{A_n} = \mathscr{K}_{\omega_n}$ for all n . Hence:

$$c_2\{z \in Q;\ \tilde{v} - \hat{v} < 0\} \leq \lim_{n\to\infty} c_2(A_n) = \lim_{n\to\infty} c_2(\omega_n) = 0 \ .$$

Remark. The above property of the elements of $\mathscr{W}$ is a fundamental tool in the study of the structure of parabolic potentials as well as in the resolution of associated variational inequalities (see [9]).

Appendix (Communication of L. Tartar) (see Lemma 2).

Proposition. Given Ω a regular bounded set of $\mathbb{R}^N$, for $\mathscr{W} = \{v \in L^2(0,T;H_0^1(\Omega));\ \frac{\partial v}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\}$, there does not exist any (continuous) function $C[\cdot]:[0,\infty) \to [0,\infty)$ such that

$$(15) \qquad \left\|\frac{\partial}{\partial t}|v|\right\|_{L^2(0,T;H^{-1})} \leq C[\|v\|_{\mathscr{W}}].$$

Proof. Let $a \in H_0^1(\Omega)$ and $f_n \in W^{1,2}(0,1)$ with

$$f_n \geq 0 \ , \ \|f_n'\|_{L^2(0,1)} = 1 \ , \quad f_n \text{ converges in } L^2(0,1) \text{ to } 0 \text{ when } n \text{ goes to } \infty \ .$$

(Take for instance $f_n(t) = \frac{\lambda}{n}[1 + \sin n\pi t]$ with $\lambda = \sqrt{2}/\pi$).

Now, applying (15) to $v_n(t) = f_n(t)a$, since $|v_n| = f_n|a|$, one would have:

$$\||a|\|_{H^{-1}(\Omega)} \leq C(\|f_n\|_{L^2(0,1)} \cdot \|a\|_{H_0^1} + \|a\|_{H^{-1}(\Omega)}) \ .$$

That is

$$\||a|\|_{H^{-1}(\Omega)} \leq C(\|a\|_{H^{-1}(\Omega)}) \ .$$

which is not true. (If $\Omega = (0,\pi)$ take for instance $a_n(x) = n \sin n x$).

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We prove here that, given an open subset Ω of $\mathbb{R}^N$, the usual parabolic capacity on $[0,T[\times \Omega$ associated with the heat operator $\frac{\partial}{\partial t} - \Delta$ can be defined using only the Hilbert norm of the space

$$W = \{v \in L^2(0,T;H_0^1(\Omega));\ \frac{\partial v}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\}.$$

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